# ON THE THEORY OF ELASTICITY OF A NONHOMOGENEOUS MEDIUM

## (K TEORII UPRUGOSTI NEODNORODNOI SREDY)

PMM Vol.28, № 4, 1964, pp.601-611

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(Received April 7, 1964)

The equations of equilibrium of an isotropic nonhomogeneous medium in which the elastic moduli are nonconstant, differentiable functions of position and Poisson's ratio $\nu$ has a constant value, have been studied by making use of the methods of separation of variables and integral transforms. The equation for the stress functions in isothermal coordinates has been given together with some applications. In the three-dimensional problem, conditions have been found for the existence of radial stress distributions. In particular, the case where the elastic modulus depends on a power of one of the Cartesian coordinates has been investigated. In this case, it has been established that the fundamental functions for the two-dimensional problem of a strip are certain confluent hypergeometric functions. The application of the Fourier method has been studied for three-dimensional problems — mainly in the case of axial symmetry. A method has been given for the numerical solution of the Boussinesq problem in terms of the familiar Flamant problem. In the particular case of a power law, when the Flamant problem has an exact solution, it turns out that the Boussinesq problem has an exact solution.

1. In the plane problem for an isotropic nonhomogeneous medium with a constant Poisson's ration, when there are no body forces or thermal stresses, the equation for Airy's stress function $F$ has the form [1]

$$(1-\nu)\Delta(m\Delta F)=\frac{\partial^2 m}{\partial x^2}\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 m}{\partial y^2}\frac{\partial^2 F}{\partial x^2}-2\,\frac{\partial^2 m}{\partial x\,\partial y}\,\frac{\partial^2 F}{\partial x\,\partial y} \qquad (1.1)$$

$$m=\frac{1}{2\mu},\qquad \mu=G,\qquad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

It can be deduced by substituting the expressions for the strains

$$e_{xx}=\frac{\partial u}{\partial x}=m\left[-\frac{\partial^2 F}{\partial x^2}+(1-\nu)\Delta F\right],\qquad e_{yy}=\frac{\partial v}{\partial y}=m\left[-\frac{\partial^2 F}{\partial y^2}+(1-\nu)\Delta F\right]$$

$$e_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=-2m\,\frac{\partial^2 F}{\partial x\,\partial y} \qquad (1.2)$$

into the compatibility condition. The notation (1.1) relates to a state of plane strain. In order to transfer to a state of plane stress it is necessary to change $\nu$ into $\nu^*=\nu/(1+\nu)$ . This equation is the Euler equation in the variational problem for the functional

$$J = \iint m \left[\left(\frac{\partial^2 F}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} + \frac{1}{2}(1-\nu)(\triangle F)^2\right] dx\, dy \qquad (1.3)$$

As was first pointed out by Castigliano and can be verified by direct calculation, the expression on the right-hand side of (1.1) is the joint invariant of two stress tensors defined by the functions $F$ and $m$ ($F$ defines the true stresses and $m$ the fictitious stresses).

Therefore, in the isothermal orthogonal coordinates

$$\xi + i\eta = \zeta, \qquad \zeta = f(z), \qquad \left|\frac{d\zeta}{dz}\right| = h, \qquad \triangle = h^2\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)$$

Equation (1.1) assumes the form

$$(1-\nu)\,\triangle\,(m\triangle F) = D_{\xi\xi} m D_{\eta\eta} F + D_{\eta\eta} m D_{\xi\xi} F - 2D_{\xi\eta} m D_{\xi\eta} F \qquad (1.4)$$

where $D$ is an operator defined by the equations

$$D_{\xi\xi} = h^2\frac{\partial^2}{\partial \xi^2} + h\left(\frac{\partial h}{\partial \xi}\frac{\partial}{\partial \xi} - \frac{\partial h}{\partial \eta}\frac{\partial}{\partial \eta}\right), \quad D_{\eta\eta} = h^2\frac{\partial^2}{\partial \eta^2} + h\left(\frac{\partial h}{\partial \eta}\frac{\partial}{\partial \eta} - \frac{\partial h}{\partial \xi}\frac{\partial}{\partial \xi}\right)$$

$$D_{\xi\eta} = h^2\frac{\partial^2}{\partial \xi\, \partial \eta} + h\left(\frac{\partial h}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial h}{\partial \eta}\frac{\partial}{\partial \xi}\right) \qquad (1.5)$$

For the stresses we have [2]

$$\sigma_\eta = D_{\xi\xi}F, \qquad \sigma_\xi = D_{\eta\eta}F, \qquad \tau_{\xi\eta} = -D_{\xi\eta}F \qquad (1.6)$$

In the polar coordinates $\xi = \ln r$ and $\eta = \varphi$ we have

$$(1-\nu)e^{2\xi}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)\left[e^{-2\xi} m\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right)F\right] = \left(\frac{\partial^2 m}{\partial \eta^2} + \frac{\partial m}{\partial \xi}\right)\left(\frac{\partial^2 F}{\partial \xi^2} - \frac{\partial F}{\partial \xi}\right) +$$

$$+\left(\frac{\partial^2 m}{\partial \xi^2} - \frac{\partial m}{\partial \xi}\right)\left(\frac{\partial^2 F}{\partial \eta^2} + \frac{\partial F}{\partial \xi}\right) - 2\left(\frac{\partial^2 m}{\partial \xi\, \partial \eta} - \frac{\partial m}{\partial \eta}\right)\left(\frac{\partial^2 F}{\partial \xi\, \partial \eta} - \frac{\partial F}{\partial \eta}\right) \qquad (1.7)$$

If in Equations (1.1) and (1.7) we set

$$m = m_0 e^{\alpha x + \beta y} \qquad (1.8)$$

$$m = m_0 e^{\alpha\xi + \beta\eta} = m_0 r^\alpha e^{\beta\varphi} \qquad (1.9)$$

an equation with constant coefficients is obtained that significantly simplifies the formulation and solution of problems.

Certain problems with a modulus as in (1.8) have already been solved [3]. In this case, separation of variables cannot be applied to Equation (1.4).

**2.** The theory of elasticity of nonhomogeneous media introduces the problem of finding the distribution of the material constants which admits the given state of stress. Such problem was first posed and solved by Lekhnitskii [4] for a state of radial stress. In the cited paper the problem was solved without use of a stress function. The obtained solution is not general in the sense of the theory of partial differential equations. We will give a solution that is general in the above-mentioned sense. We have

$$\sigma_\varphi = \tau_{r\varphi} = 0, \qquad F = r\Phi(\varphi), \qquad \sigma_r = \frac{1}{r}(\Phi + \Phi'') \tag{2.1}$$

Under these conditions, Equation (1.6) assumes the form

$$\triangle\left(\frac{\Psi}{r}\right) = \frac{1}{(1-\nu)\,r}\frac{\partial^2\Psi}{\partial r^2}, \qquad \Psi = (\Phi + \Phi'')\,m \tag{2.2}$$

This equation has a closed solution with arbitrary parameters

$$\Psi = r^\alpha (A\cos n\varphi + B\sin n\varphi)$$

$$n = n(\alpha) = \sqrt{(1-\alpha)\,[1 + \alpha\nu/(1-\nu)]} \tag{2.3}$$

Consequently, the more general solution with arbitrary functions can be represented by the Stieltjes integral

$$\Psi = \int r^\alpha\,[\cos\varphi n(\alpha)\,df(\alpha) + \sin\varphi n(\alpha)\,dg(\alpha)] \tag{2.4}$$

where $f(\alpha)$ and $g(\alpha)$ are arbitrary functions. The solution given in [4] can then be obtained by making $f(\alpha)$ and $g(\alpha)$ step-functions. In the same paper it has been shown that a power-law dependence of the elastic modulus $\mu = Ky^k$ on the Cartesian coordinate $y$ is included in this class of solutions. It can be obtained when $\alpha = -k$ in Equation (2.3). In the cited paper no formulas were given for the displacements. Here, we will deduce the corresponding results for a half-space when a concentrated normal force $P$ is applied to its boundary (the Flamant problem)

$$\sigma_r = -CPr^{-1}(\sin\varphi)^k\cos q\,(^1/_2\pi - \varphi) \tag{2.5}$$

$$C = \left[\int_0^\pi (\sin\varphi)^{k+1}\cos q\left(\frac{\pi}{2} - \varphi\right)d\varphi\right]^{-1} = \tag{2.6}$$

$$= \frac{2^{1+k}\Gamma\,[1 + {}^1/_2(1+k+q)]\,\Gamma\,[1 + {}^1/_2(1+k-q)]}{\pi\Gamma(2+k)}$$

$$u_r = \frac{(1-\nu)\,CP}{2Kk}\frac{1}{r^k}\cos q\left(\frac{1}{2}\pi - \varphi\right), \quad u_\varphi = \frac{(1-\nu)\,qCP}{2Kk\,(1+k)}\frac{1}{r^k}\sin q\left(\frac{1}{2}\pi - \varphi\right) \tag{2.7}$$

$$q = \sqrt{(1+k)\,[1 - k\nu/(1-\nu)]}$$

Consequently, when $y = 0$ the vertical displacement is

$$v(x) = \frac{\vartheta(\nu, k)}{k}|x|^{-k}, \qquad \vartheta(\nu, k) = \frac{(1-\nu)\,qCP}{2K(1+k)}\sin\frac{\pi q}{2} \tag{2.8}$$

The formulation of the problem of the action of a distributed loading $p(x)$ is meaningful provided $0 \leqslant k < 1$. Then

$$v(x) = \frac{\vartheta(\nu, k)}{k}\int p(\xi)\,|x - \xi|^{-k}d\xi \tag{2.9}$$

where the integral is extended over the whole loading section. When $k = 0$, the elastic modulus becomes constant, $\mu = K$, and the equation turns into the equation of the contact problem for a homogeneous half-space [5].

$$v(x) = \vartheta \int p(\xi) \ln \frac{1}{|x-\xi|} d\xi + \text{const}, \qquad \vartheta = \vartheta(\nu, 0) = \frac{2(1-\nu^2)}{\pi E}$$

By calculating in the same way the displacements produced by a shearing load applied to the boundary, in place of the single equation (2.9), we obtain a system of equations for the distribution of the normal and shear loadings $p(\xi)$ and $t(\xi)$. This was done by Galin [6], where it was also proved that the solution of the problem for a point force is unique. This is important, because in this case the boundary $y = 0$ is a line of singularity.

**3.** In the present case, the occurance of the singularity has been connected with the fact that such a nonhomogeneous medium cannot be physically attained. This same circumstance also implies other consequences, i.e. bounds for the possible values of the index $k$ in the power law. This is an obstacle to its application as an interpolation formula for designing a foundation. In the work of the reactive prèssure the fundamental contribution arises in layers near the surface, but precisely in this region the interpolation formula agrees badly with reality. At the same time the power-law formula is one of the most simple along with the exponential one. Therefore, there is interest in the equilibrium of an elastic strip in which the dependence of nonhomogeneity on depth can be represented by a power law with the line $y = 0$ lying outside or on the edge of the strip where it becomes infinitely rigid on passing into the half-space. We will show this does not imply a significant complication of the problem in comparison with the problem for the half-space consisting of homogeneous layers or with a nonhomogeneity that can be described in terms of an exponential law. In fact, when $m = Cy^{-k}$, Equation (1.1) gives

$$\triangle(y^{-k}\triangle F) = \frac{k(1+k)}{1-\nu} y^{-k-2} \frac{\partial^2 F}{\partial x^2} \tag{3.1}$$

Representing $F(x,y)$ by the Fourier integral

$$F(x, y) = \int_{-\infty}^{\infty} e^{i\xi x} f(y, \xi)\, d\xi \tag{3.2}$$

simple calculations yield (3.3)

$$y^2 \left(\frac{d^2}{dy^2} - \xi^2\right)^2 - 2ky\left(\frac{d^2}{dy^2} - \xi^2\right)\frac{df}{dy} + k(1+k)\frac{d^2 f}{dy^2} + h\xi^2 f = 0 \qquad \left(h = \frac{\nu k(1+k)}{1-\nu}\right)$$

This equation can be written in dimensionless form by setting $\eta = \xi y$. Then we have

$$\eta^2\left(\frac{d^2}{d\eta^2} - 1\right) f - 2k\eta\left(\frac{d^3}{d\eta^3} - \frac{d}{d\eta}\right) f + k(1+k)\frac{d^2 f}{d\eta^2} + hf = 0 \tag{3.4}$$

We will represent the solution of this equation by the Laplace transform

$$f(\eta) = \int e^{\eta t} \varphi(t)\, dt \tag{3.5}$$

taken over a suitably chosen contour [7]. After integration by parts and reductions we obtain Equation

$$(t^2-1)^2\frac{d^2\varphi}{dt^2}+2(k+4)(t^2-1)t\frac{d\varphi}{dt}+ \\ +[(k+3)(k+4)+h-2k-4]\varphi=0 \tag{3.6}$$

This equation has regular singularities at the points $t=-1, 1, \infty$. Calculation of the characteristic exponents yields the scheme of the general solution

$$\varphi(t)=P\begin{Bmatrix} 1 & -1 & \infty \\ -{}^1/_2(3+k-q) & -{}^1/_2(3+k-q) & 3+k, & t \\ -{}^1/_2(3+k+q) & -{}^1/_2(3+k+q) & 4+k \end{Bmatrix}$$

The substitution $2u=1-t$ brings the elements in the upper row into the standard form $(0, 1, \infty)$. As independent particular solutions we can take the following hypergeometric functions

$$\begin{aligned} \varphi_1 &= [u(1-u)]^{-{}^1/_2(3+k)+{}^1/_2q}\,{}_2F_1(q, q+1; u) \\ \varphi_2 &= [u(1-u)]^{-{}^1/_2(3+k)+{}^1/_2q}u^{-q}\,{}_2F_1(0, 1; 1-q; u) \end{aligned} \tag{3.7}$$

which, of course, can be transformed into the elements

$$\varphi_1=u^{-{}^1/_2(3+k)+{}^1/_2q}(1-u)^{-{}^1/_2(3+k)-{}^1/_2q}, \quad \varphi_2=u^{-{}^1/_2(3+k)-{}^1/_2q}(1-u)^{-{}^1/_2(3+k)+{}^1/_2q} \tag{3.8}$$

Introducing these expressions into the Laplace transform (3.5), we find that the independent particular solutions of (3.4) are [7] the confluent hypergeometric functions $\Phi(a; c; x)$, $\Psi(a; c; x)$, where $a=-\frac{1}{2}(1+k)+\frac{1}{2}q$, $c=-(1+k)$, $x=2\eta$ with a supplementary factor $e^{-\eta}$. The general solution has the form

$$\begin{aligned} f(\eta)=e^{-\eta}\{&C_1\Phi(-{}^1/_2(1+k)+{}^1/_2q; -(1+k); 2\eta)+ \\ &+C_2\Psi(-{}^1/_2(1+k)+{}^1/_2q; -(1+k); 2\eta)+ \\ &+C_3\Phi(-{}^1/_2(1+k)-{}^1/_2q; -(1+k); 2\eta)+ \\ &+C_4\Psi(-{}^1/_2(1+k)-{}^1/_2q; -(1+k); 2\eta)\} \end{aligned} \tag{3.9}$$

or, changing to Whittaker functions,

$$\begin{aligned} f(\eta)=\eta^{{}^1/_2+{}^1/_2k}\{&C_1M_{-{}^1/_2q,\,-1-{}^1/_2k}(2\eta)+C_2W_{-{}^1/_2q,\,-1-{}^1/_2k}(2\eta)+ \\ &+C_3M_{{}^1/_2q,\,-1-{}^1/_2k}(2\eta)+C_4W_{{}^1/_2q,\,-1-{}^1/_2k}(2\eta)\} \end{aligned} \tag{3.10}$$

In the theory of integral transforms with Whittaker functions, a certain amount of progress has been made recently (see [8]). Therefore, we have the opportunity of applying the methods of integral transforms both to the study of the state of stress in plates and to the study of related models of elastic foundations. Here we will consider one example in which the first fundamental problem for a strip on a rigid foundation under the action of normal loading on the upper boundary has a relatively simple solution.

4. We will assume that $m = cy$ inside the strip $0 < y < H$, and $m = 0$ for $y < 0$. The case of an exponent $k = -1$ is singular since the method considered above then gives only two independent particular solutions of equation (3.3). However, in the present case this equation can be solved directly. By reduction of order, together with the required regularity of $f(y,\xi)$ at $y = 0$, we have

$$\frac{d^2 f}{dy^2} - \xi^2 f = A\frac{\sinh \xi y}{y} \tag{4.1}$$

For the strains and stresses from Formulas (1.2) and (3.2), it follows that

$$\frac{\partial u}{\partial x} = cy \int\limits_{-\infty}^{\infty} \left[\xi^2 + (1-\nu)\left(\frac{d^2}{dy^2} - \xi^2\right)\right] f(y, \xi)\, e^{i\xi x} d\xi$$

$$\frac{\partial v}{\partial y} = cy \int\limits_{-\infty}^{\infty} \left[-\frac{d^2}{dy^2} + (1-\nu)\left(\frac{d^2}{dy^2} - \xi^2\right)\right] f(y, \xi)\, e^{i\xi x} d\xi \tag{4.2}$$

$$\sigma_y = -\int\limits_{-\infty}^{\infty} \xi^2 f(y, \xi)\, e^{i\xi x} d\xi, \qquad \sigma_x = \int\limits_{-\infty}^{\infty} \frac{d^2 f}{dy^2}\, e^{i\xi x} d\xi, \qquad \tau_{xy} = -i\int \xi \frac{df}{dy}\, e^{i\xi x} d\xi \tag{4.3}$$

Hence we conclude that when $y = 0$ we have only a rigid displacement that we assume to be zero and that on the boundaries $y = 0$, $y = H$ there will be no shear stress when $f_y'(0, \xi) = f_y'(H, \xi) = 0$. Hence it follows that

$$f(H, \xi) = \frac{1}{2\pi\xi^2} \int\limits_{-\infty}^{\infty} e^{-i\xi t} p(t)\, dt \tag{4.4}$$

The solution of Equation (4.1) satisfying these conditions has the form

$$f(y, \xi) = \frac{g(y, \xi)}{2\pi\xi^2} \int\limits_{-\infty}^{\infty} e^{-i\xi t} p(t)\, dt \tag{4.5}$$

where

$$g(y, \xi) = \left[\int\limits_0^H \frac{\cosh \xi\alpha \sinh \xi\alpha}{\alpha}\, d\alpha\right]^{-1} \int\limits_0^H K(y, \xi; \alpha)\frac{\sinh \xi\alpha}{\alpha}\, d\alpha \tag{4.6}$$

$$K(y, \xi; \alpha) = \begin{cases} \cosh \xi (H - y) \cosh \xi\alpha, & \alpha < y \\ \cosh \xi (H - \alpha) \cosh \xi y, & \alpha > y \end{cases} \tag{4.7}$$

The exponential integrals occuring in this solution are tabulated functions and there is no obstacle here in obtaining numerical results.

5. Before passing to the consideration of three-dimensional problems, we will establish a result of a negative character, namely that in a three-dimensional nonhomogeneous medium it is in general impossible to have a radial distribution of stresses. More precisely this means that the Lekhnitskii problem does not have a solution in arbitrary functions like (2.4). There is only one particular solution containing one arbitrary function. When the elastic modulus depends on only one coordinate, this solution is identical with that obtained in [9].

Here it is convenient to pass to spherical coordinates. The equilibrium equations reduce to the single equation

$$\frac{\partial \sigma_r}{\partial r} + \frac{2}{s}\sigma_r = 0 \tag{5.1}$$

Hence, by virtue of the axial symmetry of the problem, it follows that

$$\sigma_r = \sigma = r^{-2} S(\theta) \tag{5.2}$$

For the strains we have

$$e_{rr} = \varepsilon, \qquad e_{\theta\theta} = e_{\varphi\varphi} = -\nu\varepsilon, \qquad e_{r\theta} = e_{\theta\varphi} = e_{\varphi r} = 0 \qquad (\varepsilon = \sigma / E) \tag{5.3}$$

We will not write out the compatibility conditions in spherical coordinates [10] but will quote the result of substituting expressions (5.3) into them

$$\left(\nu \frac{\partial^2 \varepsilon}{\partial \theta^2} + \cot\theta \frac{\partial \varepsilon}{\partial \theta} + 2r \frac{\partial \varepsilon}{\partial r}\right) + 2(1+\nu)\varepsilon = 0, \qquad \frac{\partial \varepsilon}{\partial \theta} + \nu r \frac{\partial^2 \varepsilon}{\partial r \partial \theta} = 0 \tag{5.4}$$

$$\nu r^2 \frac{\partial^2}{\partial r^2}(r\varepsilon) + r\frac{\partial \varepsilon}{\partial r} - \cot\theta \frac{\partial \varepsilon}{\partial \theta} = 0, \qquad \nu r^2 \frac{\partial^2}{\partial r^2}(r\varepsilon) + r\frac{\partial \varepsilon}{\partial r} - \frac{\partial^2 \varepsilon}{\partial \theta^2} = 0$$

From the last two conditions it follows that

$$\frac{\partial^2 \varepsilon}{\partial \theta^2} - \cot\theta \frac{\partial \varepsilon}{\partial \theta} = 0, \qquad \varepsilon = f(r)\cos\theta + g(r) \tag{5.5}$$

Substitution of this result into the second condition (5.4) yields

$$f(r) + \nu r f'(r) = 0, \qquad f(r) = A r^{-1/\nu} \tag{5.6}$$

Substitution into the first condition leads to the equations

$$(\nu + 1) g(r) + \nu r g'(r) = 0, \qquad g(r) = B r^{-1-1/\nu} \tag{5.7}$$

Thus

$$\varepsilon = A r^{-1/\nu}\cos\theta + B r^{-1-1/\nu} \tag{5.8}$$

This function satisfies the last equations (5.4) identically.

Hence, for the elastic modulus we have

$$E = \frac{\sigma}{\varepsilon} = \frac{r^{1/\nu - 1} S(\theta)}{Ar\cos\theta + B} \tag{5.9}$$

If $E$ is a function of $z$ only, then

$$S(\theta) = C(\cos\theta)^{1/\nu - 1}, \qquad E = \frac{A r^{1/\nu - 1}}{Az + B} \tag{5.10}$$

When $A = 0$ we have one, and when $B = 0$ we have another of the solutions deduced in [10]. For the displacements we have

$$u_r = A\frac{\nu}{\nu - 1} r^{1-1/\nu}\cos\theta - \nu B r^{-1/\nu}, \quad u_\theta = \frac{-\nu^2}{\nu - 1} A r^{1-1/\nu}\sin\theta \tag{5.11}$$

(the rigid displacements has been omitted). On the surface $z = 0$, $\theta = \frac{1}{2}\pi$ they are

$$u_r = -\nu B r^{-1/\nu}, \qquad u_\theta = \frac{\nu^2}{\nu - 1} A r^{1-1/\nu} \tag{5.12}$$

In order that this solution is meaningful in the case of a distributed load, it is necessary that the indices in these formulas are greater than $-2$. If $B \neq 0$, this requires $\nu > \frac{1}{2}$, which is physically unrealistic. When $B = 0$, it is necessary that $\nu > \frac{1}{3}$. Thus, actually

$$E = z^{1/\nu-2} F(\theta), \qquad S(\theta) = cF(\theta)(\cos\theta)^{1/\nu-1} \qquad (1/2 > \nu > 1/3) \qquad (5.13)$$

When $F(\theta) = \text{const}$, we obtain the solution found in [10].

**6.** The equilibrium equation for the displacements in a nonhomogeneous medium can be obtained in the usual way from the Cauchy equations by substituting into them the expressions for the stresses in terms of the strains. Introducing the notation

$$\theta = \operatorname{div} \mathbf{u}, \qquad \epsilon = \operatorname{def} \mathbf{u}$$

the stress tensor is given by Formula

$$\mathbf{S} = \lambda\theta\mathbf{I} + 2\mu\epsilon \qquad (\mathbf{I} \text{ is the unit tensor}) \qquad (6.1)$$

and the equilibrium equation for the displacements has the form

$$(\lambda + \mu)\nabla\theta + \mu\triangle\mathbf{u} + \theta\nabla\lambda + 2\epsilon\cdot\nabla\mu = 0$$

where

$$\triangle = \operatorname{grad}\operatorname{div} - \operatorname{rot}\operatorname{rot} \qquad (6.2)$$

When $\nu = \text{const}$, this equation simplifies to

$$\alpha\nabla\theta + \triangle u + [(\alpha - 1)\theta I + 2\epsilon]\cdot\nabla\ln\mu = 0 \qquad \left(\alpha = \frac{1}{1-2\nu}\right) \qquad (6.3)$$

We consider a half-space with the elastic modulus depending on the depth, $\mu = \mu(z)$. In Cartesian coordinates, we have

$$\alpha\frac{\partial\theta}{\partial x} + \triangle u + q(z)\, e_{zx} = 0, \qquad \alpha\frac{\partial\theta}{\partial y} + \triangle v + q(z)\, e_{zy} = 0$$
$$\alpha\frac{\partial\theta}{\partial z} + \triangle w + q(z)\,[(\alpha-1)\theta + 2e_{zz}] = 0, \qquad q(z) = \frac{d}{dz}\ln\mu(z) \qquad (6.4)$$

If $\mu$ depends on the depth exponentially, then $q(z) = \text{const}$, and Equation (6.4) reduces to a system with constant coefficients. This case will not be considered. Other related problems have already been considered [11]. Below we will treat the basic case when $1/q(z)$ has the form $az + b$, or, by translating the coordinate origin, more simply $z/k$, i.e. when there is a power-law dependence of the elastic modulus on the depth, $\mu = Kz^k$; here $\mu = a + bz$ is a special case. We will seek the solution of equation (6.4) in the form of the Fourier integral transform

$$(u, v, w) = \iint_{-\infty}^{\infty} (U, V, W)\, e^{i(\xi x + \eta y)}\, d\xi d\eta \qquad (6.5)$$

In the general case, this leads to the sixth-order system

$$\alpha\xi\left(i\frac{dW}{dz} - \xi U - \eta V\right) + \left(\frac{d^2U}{dz^2} - \rho^2 U\right) + q(z)\left(\frac{dU}{dz} + i\xi W\right) = 0 \qquad (6.6)$$

$$\alpha\eta\left(i\frac{dW}{dz}-\xi U-\eta V\right)+\left(\frac{d^2V}{dz^2}-\rho^2V\right)+q(z)\left(\frac{dV}{dz}+i\eta W\right)=0 \qquad \text{(6.6) cont.}$$

$$\alpha\left[\frac{d^2W}{dz^2}+i\left(\xi\frac{dU}{dz}+\eta\frac{dV}{dz}\right)\right]+\left(\frac{d^2W}{dz^2}-\rho^2W\right)+$$
$$+q(z)\left[(\alpha+1)\frac{dW}{dz}+(\alpha-1)i(\xi U+\eta V)\right]=0 \qquad (\rho^2=\xi^2+\eta^2)$$

The study of this system in the general case of an analytic coefficient $q(z)$ is difficult. In the case of power law, a Laplace transformation makes it possible to reduce the order of the system by two and to eliminate the regular singularity at infinity. Assuming in this case

$$(U, V, W)=\int(\Phi, \mathrm{X}, \Psi)\, e^{zt}dt \tag{6.7}$$

after some calculation we arrive at a system containing the first time derivatives of the functions $\Phi$, $\mathrm{X}$, $\Psi$. In order to avoid an awkward notation, we will present this system in matrix form

$$\frac{d}{dt}\begin{bmatrix}\Phi\\ \mathrm{X}\\ \Psi\end{bmatrix}=\begin{bmatrix}t^2-\rho^2-\alpha^2\xi^2 & -\alpha\xi & i\alpha t\xi\\ -\alpha\xi\eta & t^2-\rho^2-\alpha^2\eta^2 & i\alpha t\eta\\ i\alpha t\xi & i\alpha t\eta & (\alpha+1)t^2-\rho^2\end{bmatrix}^{-1}\times$$
$$\times\begin{bmatrix}(k-2)t & 0 & (k-\alpha)i\xi\\ 0 & (k-2)t & (k-\alpha)i\eta\\ (k\alpha-k-\alpha)i\xi & (k\alpha-k-\alpha)i\eta & (\alpha+1)(k-2)t\end{bmatrix}\times\begin{bmatrix}\Phi\\ \mathrm{X}\\ \Psi\end{bmatrix} \tag{6.8}$$

The determinant of the first matrix is

$$\triangle(t)=(\alpha+1)(t^2-\rho^2)^2$$

For this system the point at infinity is a regular point. Nevertheless, the analytical investigation and the construction of solutions is a complex and difficult problem.

**7.** The situation is somewhat simplified in the axisymmetric problem. In this case, the system of equations for the displacements reduces to two independent subsystems, one of which describes a state of spherical symmetry

$$\alpha\frac{\partial\theta}{\partial r}+\triangle u-\frac{u}{r^2}+q(z)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial r}\right)=0$$
$$\alpha\frac{\partial\theta}{\partial z}+\triangle w+q(z)\left[(\alpha-1)\theta+2\frac{\partial w}{\partial z}\right]=0 \tag{7.1}$$
$$u=u_r, \qquad w=u_z, \qquad \theta=\frac{\partial u}{\partial r}+\frac{u}{\partial r}+\frac{\partial w}{\partial z}, \qquad \triangle=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}$$

and the other a state of torsion

$$\triangle v-\frac{v}{r^2}+q(z)\frac{\partial v}{\partial z}=0, \qquad v=u_\varphi \tag{7.2}$$

The last equation can be reduced to a form

$$\triangle\omega+q(z)\frac{\partial\omega}{\partial z}=0 \qquad \left(v=\frac{\partial\omega}{\partial z}\right)$$

In the case $q(z) = kz^{-1}$, $0 < k < 1$, this equation was studied in [12], where the fundamental functions (hypergeometric were determined in a system of oblate spherical coordinates. Here we will give the reduction of (7.1) to a system of first-order, ordinary differential equations (when $q(z) = kz^{-1}$). For this we will represent the displacements $u$, $w$ by the Hankel transforms

$$u = \int_0^\infty sJ_1(rs) f(z, s)\, ds, \qquad w = \int_0^\infty sJ_0(rs) g(z, s)\, ds \tag{7.3}$$

For the functions $f$ and $q$ we then obtain the system

$$\begin{gathered} \frac{d^2 f}{dz^2} - (\alpha + 1) s^2 g - \alpha s \frac{dq}{dz} + q(z)\left(\frac{df}{dz} - sg\right) = 0 \\ (\alpha + 1) \frac{d^2 g}{dz^2} - s^2 g + \alpha s \frac{df}{dz} + q(z)\left[(\alpha - 1) sf + (\alpha + 1) \frac{dq}{dz}\right] = 0 \end{gathered} \tag{7.4}$$

By setting $q(z) = kz^{-1}$ and making the substitution $\zeta = zs$, we obtain

$$\begin{gathered} \zeta\left[\alpha \frac{df}{d\zeta} + (\alpha + 1) \frac{d^2 g}{d\zeta^2} - g\right] + k\left[(\alpha - 1) f + (\alpha + 1) \frac{dg}{d\zeta}\right] = 0 \\ \zeta\left[\frac{d^2 f}{d\zeta^2} - (\alpha + 1) g - \alpha \frac{dg}{d\zeta}\right] + k\left(\frac{df}{d\zeta} - g\right) = 0 \end{gathered} \tag{7.5}$$

We now reduce the order of the system by means of the Laplace transforms

$$f(\zeta) = \int \varphi(t)\, e^{\zeta t} dt, \qquad g(\zeta) = \int \psi(t)\, e^{\zeta t} dt \tag{7.6}$$

This yields

$$\frac{d}{dt}\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} a_{11}(t), & a_{12}(t) \\ a_{21}(t), & a_{22}(t) \end{bmatrix} \times \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \tag{7.7}$$

$$a_{ij} = \frac{A_{ij}(t)}{\Delta(t)}, \qquad \Delta(t) = (\alpha + 1)(t^2 - 1)^2 \tag{7.8}$$

$$\begin{gathered} A_{11} = t\,[(\alpha + 1)(k - 2) t^2 - \alpha(k\alpha - k - \alpha) - (k - 2)] \\ A_{12} = (\alpha + 1)(k\alpha - k - \alpha) t^2 - (\alpha - k) \\ A_{21} = (\alpha - k) t^2 - (\alpha + 1)(k\alpha - k - \alpha) \\ A_{22} = t\,[(\alpha + 1)(k - 2) t^2 - (\alpha + 1)^2 (k - 2) - \alpha(\alpha - k)] \end{gathered} \tag{7.9}$$

As can be seen, all the singular points of this system are regular. By eliminating one function, we obtain for the other a second-order equation. A quadratic transformation then reduces the latter to the Heun equation with four singular points (of these, one is an apparent singularity).

We point out that in solving the plane-strain problem by the method of Fourier transforms (in the $z$-coordinate) and by the Laplace transform (in the $y$-coordinate), one is lead to the same system of equations (7.7). At the same time, in the plane problem the analytical aspect is considerably simplified by using a stress function. The derivation of this function in the problem of a strip reduces to a Fourier transform of a linear combination of Whittaker functions, whereas the displacements have a more complicated analytical structure as can be seen from Formulas (1.2). Thus, in the axisymmetric problem one is compelled to seek another method that is more effective from the analytical point of view. Here also a stress function might be introduced, but it is more rational to make use of the relations between plane and axisymmetric problems as established by Mossakovskii [13] and Aleksandrov [14] for a homogeneous medium. Since these relations are represented by integral transforms that do not affect the $z$-coordinate, they remain valid also in the case of nonhomogeneity with respect to this coordinate (*). In the last section we will give an application of this method to the problem of calculating the normal displacements on the boundary of a half-space produced by a concentrated normal force applied to the boundary (the Boussinesq problem).

**8.** Since the shear stresses on the boundary vanish, the present problem can be handled with the two formulas [14]

$$W(r) = \int_{-r}^{+r} w^-(x) \frac{dx}{\sqrt{x^2 - r^2}}, \qquad P^-(t) = \int_0^t p^-(x)\,dx = \frac{1}{\pi}\int_0^t \frac{p(r)\,r dr}{\sqrt{t^2 - r^2}} \tag{8.1}$$

Let $p(r)$ be a piece-wise integrable, bounded function. (The dash superscripts denote values for the plane problem).

Then $P^-(t) = O(t^{-1})$ as $t \to \infty$. Bearing this in mind, for the plane problem the displacements are given by

$$w^-(x) = \int_{-\infty}^{\infty} p^-(t)\, K^-(x, t)\, dt = -\int P^-(t) \frac{\partial K^-}{\partial t} dt \tag{8.2}$$

Let

$$K^-(x, t) = K^-(|x - t|), \qquad K^-(x) = \int_{-\infty}^{\infty} K_*^-(s)\, e^{isx} ds \tag{8.3}$$

In view of the remarks regarding $P^-(t)$, we can reverse the order of integration in (8.2) after (8.3) has been substituted into it. Thus

$$w^-(x) = \int_{-\infty}^{\infty} isK_*^-(s)\, e^{isx} ds \int_{-\infty}^{\infty} e^{-ist} P^-(t)\, dt \tag{8.4}$$

and from the first formula of (8.1) follows

$$w(r) = \int_{-\infty}^{\infty} i\pi s K_*^-(s)\, J_0(rs)\, ds \int_{-\infty}^{\infty} e^{-ist} P^-(t)\, dt \tag{8.5}$$

The result of substituting the second formula (8.1) into the inner integral (8.5) can be obtained after overcoming some complications. However, it will not be necessary to do this. For the concentrated force $P$ it is easy to find that

$$P^-(t) = (2\pi^2)^{-1} P t^{-1}$$

and the inner integral is equal to $-i\pi \operatorname{sgn} s$. In this case we find

$$w(r) = P \int_0^{\infty} sK_*^-(s)\, J_0(rs)\, ds \tag{8.6}$$

For a unit force $P = 1$ the displacement $w(r) = K(r)$ is the kernel of the functional for the displacements on the boundary of a half-space (layer)

$$w(x, y) = \iint p(\xi, \eta)\, K(r)\, d\xi d\eta, \qquad r = \sqrt{(x - \xi)^2 + (y - \eta)^2} \tag{8.7}$$

Inverting (8.3) and inserting the result into (8.6), we obtain a direct

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*) The author thanks V.I. Mossakovskii for suggesting this idea.

representation of the kernel $K(r)$ in terms of the kernel $K^-(x)$ of the symmetric plane problem

$$K(r) = \frac{-1}{\pi}\int_r^\infty K^{-\prime}(x)\,\frac{dx}{\sqrt{x^2 - r^2}} \qquad (8.8)$$

When applied to the power-law kernel (2.8), this gives

$$K(r) = \frac{1}{2\sqrt{\pi}}\,\frac{\Gamma(1/2 + 1/2k)}{\Gamma(1 + 1/2k)}\,\vartheta(\nu, k)\,r^{-1-k} \qquad (8.9)$$

Hence, when $k = 0$ or $\nu = 1/(2+k)$ one can derive already known results [5, 10 and 12].

N o t e 1 . The direct calculation of $K_*^-(s)$ from (8.3) sometimes leads to the calculation of the integrals (8.6) between the terminals of the admissible values of the parameters (as in the example treated). Since the derivation of (8.8) consists in the calculation of the convolution of a generalized Fourier transform and a Hankel transform, these integrals should be understood in a generalized sense.

N o t e 2 . In the case where the elastic modulus depends on the power of one Cartesian coordinate, the Fourier transforms of the stresses generate the linear manifold of confluent hypergeometric functions of this coordinate. The transform connecting the plane problem with with the axisymmetric problem does not alter the analytic character of this manifold. In the axisymmetric problem the Hankel transforms generate the same linear manifold. This determines the degree and character of the analytical difficulties in both the above problems with such nonhomogeneities in the elastic properties.

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EDITORIAL NOTE

*) English version by John Wiley, p.36, 1964.

Translated by D.B.McV.